The Herfindahl-Hirschman Index and the distribution of social surplus *

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Abstract

I show that in a broad range of oligopoly models where firms have (not necessarily identical) constant marginal cost, HHI is an increasing function of the ratio of producers' surplus and consumers' surplus and therefore reflects the division of surplus between firms' owners and consumers.

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1 Introduction

The Herfindahl-Hirschman Index (HHI) of concentration, calculated as the sum of the squared market shares of firms, is widely used both in research and in practice.[1] The question though is why HHI is a good measure of concentration and how should we interpret different levels of HHI and changes in HHI. In particular, is HHI related to any measures of welfare and do changes in HHI have any normative implications?

In this paper, I show that if one is willing to assume that firms have constant marginal costs, then in a broad range of oligopoly models, HHI can be expressed as an increasing function of the ratio of producers' surplus to consumers' surplus. HHI then has a clear normative interpretation as it reflects how social surplus is divided between firms' owners and consumers, with higher values of HHI being associated with outcomes that are more favorable to firms' owners and less favorable to consumers. For example, the result suggests that when policymakers set safe harbor tests based on HHI, as done for instance in merger analysis, they are willing to tolerate certain divisions of social welfare between firms' owners and consumers, but may not tolerate others that are more favorable to firms' owners and less favorable to consumers.

I begin by showing that in a Cournot model where firms have (not necessarily identical) constant marginal costs, HHI can be expressed as $H = \frac{1}{\eta(Q^*)} \frac{PS^*}{CS^*}$, where PS^* and CS^* are the equilibrium values of producers' and consumers' surplus, and $\eta(Q^*)$ is the elasticity of consumers' surplus with respect to the equilibrium output level. In other words, HHI is proportional to the ratio of producers' surplus to consumers' surplus, and the factor of proportionality is equal to the inverse of $\eta(Q^*)$. To illustrate, the 2010 horizontal merger guidelines of the DOJ and the FTC define markets as unconcentrated if HHI is below $1,500$ and state that "Mergers resulting in unconcentrated markets are unlikely to have adverse competitive effects and ordinarily require no further analysis," but define markets as highly concentrated if HHI is above $2,500$ and state that "Mergers resulting in highly concentrated markets that involve an increase in the HHI of more than

[1] The index can be viewed as a weighted sum of the market shares of firms, where the weights are equal to the market shares. The index was independently developed by Hirschman (1945), who used it as a measure of a country's foreign trade concentration, and by Herfindahl (1950), who used it to measure "gross changes" in the concentration of the U.S. steel industry. The index was then used by Stigler (1964) in his seminal paper on collusion, and became popular after William Baxter introduced it in the Department of Justice when he served as the Assistant Attorney General in charge of the Antitrust Division in the early 1980's, and especially after it was included in the 1982 horizontal merger guidelines. For a history of HHI, see Calkins (1983).

200 points will be presumed to be likely to enhance market power." If $\eta(Q^*) = 2$ (below I show that this is the case for instance when demand is linear), these rules can be interpreted as reflecting a willingness of the 2010 horizontal merger guidelines to tolerate mergers when consumers' surplus is at least 3.3 times larger than producers' surplus, but not tolerate relatively larger mergers when consumers' surplus is less than twice as large as producers' surplus.[2]

The result that $H = \frac{1}{\eta(Q^*)} \frac{PS^*}{CS^*}$ also implies that if we hold $\eta(Q^*)$ constant, an increase in HHI is associated with an increase in producers' surplus relative to consumers' surplus. It turns out that for a large class of inverse demand functions, including linear, constant elasticity, and log-linear inverse demand functions, $\eta(Q^*)$ is indeed a constant and is equal to the inverse of the cost pass-through rate. Consequently, an increase in HHI, due to demand or cost shocks or due to a decrease in the number of firms (say due to a merger or an exit), is associated with an increase in producers' surplus relative to consumers' surplus. This conclusion does not change when $\eta(Q^*)$ is not constant, provided that the product of HHI and $\eta(Q^*)$ is increasing when HHI is increasing.

Turning to differentiated products, I show that in models with a linear demand system and constant marginal costs (e.g., Spence (1976), Dixit (1979), Singh and Vives (1984), Shubik and Levitan (1980), or the Vickery-Salop circular city model (Vickery (1964) and Salop (1979)) with either quantity or price competition, HHI can be expressed as an increasing function of the ratio of producers' surplus to consumers' surplus. Hence, as in the Cournot case, larger values of HHI are associated with distributions of social surplus which are more favorable to firms' owners and less favorable to consumers. It should be noted however that since HHI is endogenous, demand or cost shocks or changes in the number of firms, which cause an increase in HHI, may also affect the relationship between HHI and $\frac{PS^*}{CS^*}$ and hence it is not immediately obvious that an increase in HHI will be always associated with an increase in $\frac{PS^*}{CS^*}$. However I show that an increase in HHI due to demand or cost shocks is always associated with an increase in $\frac{PS^*}{CS^*}$. Moreover, an increase in HHI due to a change in the number of firms is also associated with an increase in $\frac{PS^*}{CS^*}$ when demand is given by the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, or when firms are symmetric and demand is given by the Shubik and Levitan (1980) specification.

The literature has already provided several interpretations of HHI. These interpretations are based on the Cournot model.[3] Cowling and Waterson (1976) show that when each firm has

[2] To see why, note that if HHI is 1,500 points, then $0.15 = \frac{1}{2} \frac{PS^*}{CS^*}$, so $CS^* = \frac{PS^*}{2 \times 0.15} = 3.3 PS^*$ and if HHI is 2,500 points, then $0.25 = \frac{1}{2} \frac{PS^*}{CS^*}$, so $CS^* = \frac{PS^*}{2 \times 0.25} = 2PS^*$.

[3] An exception is Nocke and Schutz (2018) who study oligopoly with price competition and show, using a Taylor approximation, that HHI is proportional to the difference between consumer surplus and aggregate surplus under

a constant marginal cost, HHI equals $\varepsilon\frac{PS^*}{R^*}$, where ε is the elasticity of demand and R^* is the equilibrium aggregate revenue.[4] Dansby and Willig (1979) consider a more general setting where firms do not necessarily have constant marginal costs and show that HHI equals $(\varepsilon\phi)^2$, where ϕ is the "industry performance gradient," which reflects the rate of change in welfare as output is adjusted by moving within a fixed distance from the status quo. Kwoka (1985) considers a similar setting and shows that HHI equals εL^*, where $L^* \equiv \sum_{i=1}^{n} s_i^* L_i^*$ is a weighted average of the equilibrium Lerner indices of individual firms, with s_i^* being the equilibrium market share of firm i, and $L_i^* = \frac{p^*-c_i'}{p}$ is the equilibrium Lerner index of firm i. The three papers then imply that if we hold ε constant, an increase in HHI is associated with an increase in (i) the ratio of producers' surplus to aggregate revenues, (ii) the industry performance gradient, and (iii) the average price-cost margin in the industry. Farrell and Shapiro (1990) also consider a general Cournot model and show that an increase in HHI may be associated with an increase in welfare even when output falls. The reason is that in a Cournot equilibrium, larger firms have lower marginal costs, so if production shifts from small to large firms (and hence HHI increases), the cost savings from more efficient production may outweigh the negative effect of the reduction in total output. While these results are all helpful, they do not tell us how HHI is related to the distribution of surplus between firms' owners and consumers, which is the main focus of this paper.

The rest of the paper is organized as follows. In Section 2, I present the main result, which I establish in the context of the Cournot model. In Section 3, I show that the main insight from Section 2 generalizes to the case of differentiated products with linear demands. Concluding remarks are in Section 4. The Appendix contains some technical proofs and derivations.

2 The main result

Consider a Cournot model with n firms. The cost of each firm i is $c_i(q_i) = F_i + k_i q_i$, where $F_i > 0$ is a fixed cost, $k_i > 0$ is firm i's constant marginal cost, and q_i is firm i's output. The inverse demand function is $p(Q)$, where $Q = \sum_{i=1}^{n} q_i$ is aggregate output, and $p'(Q) < 0$ and $p'(Q) + p''(Q) q_i < 0$ for all q_i. These assumptions are standard and ensure that the model is well behaved. Each firm i

oligopoly and under monopolistic competition.

[4] If we multiply the individual first-order conditions for profit maximization, $p + p'q_i - k_i = 0$, by q_i, and sum up the product over all firms, we get $\sum_{i=1}^{n}\left(pq_i + p'(q_i)^2 - k_i q_i\right) = 0$, which can be rewritten as $\sum_{i=1}^{n}(p-k_i)q_i = -p'\sum_{i=1}^{n}(q_i)^2 \equiv -p'H$. Dividing both sides of the equation by $\sum_{i=1}^{n} pq_i = pQ$, using the fact that $\varepsilon \equiv -\frac{pQ}{p'}$, and rearranging, yields $\frac{\sum_{i=1}^{n}(pq_i - c_i)}{\sum_{i=1}^{n} pq_i} = \frac{-p'H}{pQ} \equiv \frac{H}{\varepsilon}$.

chooses its output, q_i, to maximize its respective profit

$$\pi_i = p(Q) q_i - F_i - k_i q_i.$$

An interior Nash equilibrium is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:[5]

$$p(Q) + p'(Q) q_i - k_i = 0, \qquad i = 1, 2, , \ldots, n.$$

The price-cost margin of each firm i in an interior Nash equilibrium is given by

$$p(Q^*) - k_i = -p'(Q^*) q_i^*,$$

where $Q^* = \sum_{i=1}^n q_i^*$. Using this expression, the equilibrium producer surplus of each firm i (its profit gross of fixed cost) can be written as

$$PS_i^* = (p(Q^*) - k_i) q_i^* = -p'(Q^*) (q_i^*)^2. \tag{1}$$

The equilibrium value of consumers' surplus is

$$CS^* = \int_0^{Q^*} p(z)\, dz - p(Q^*) Q^*. \tag{2}$$

Noting that $(CS^*)' = -p'(Q^*) Q^*$, and using (1), (aggregate) producers' surplus is given by

$$PS^* \equiv \sum_{i=1}^n PS_i^* = \frac{(CS^*)' \sum_{i=1}^n (q_i^*)^2}{Q^*}. \tag{3}$$

Given a Nash equilibrium, the market share of firm i is simply $\frac{q_i^*}{Q^*}$. Hence, HHI is given by

$$H = \sum_{i=1}^n \left(\frac{q_i^*}{Q^*}\right)^2 = \frac{\sum_{i=1}^n (q_i^*)^2}{(Q^*)^2}. \tag{4}$$

Substituting for $\sum_{i=1}^n (q_i^*)^2$ from (3) into (4) and rearranging, yields the following result:

Proposition 1: *In an n firms Cournot model, where firms have (possibly different) constant marginal costs,*

$$H = \frac{PS^*}{\eta(Q^*) CS^*}, \tag{5}$$

where $\eta(Q^*) \equiv \frac{Q^*(CS^*)'}{CS^*} > 0$ is the elasticity of consumers' surplus with respect to output.

[5] The equilibrium is interior if the price when the $n-1$ most efficient firms produce, is lower than k_i for the least efficient firm.

Proposition 1 implies that HHI is proportional to $\frac{PS^*}{CS^*}$, which is the ratio of producers' surplus to consumers' surplus; the factor of proportionality is the inverse of the elasticity of consumers' surplus, $\eta(Q^*)$. That is, the value of HHI reflects the division of social surplus between firms' owners and consumers. The share of consumers in the social surplus is larger the lower $\eta(Q^*)$ is, i.e., the more inelastic consumers' surplus is with respect to output.

To illustrate Proposition 1, recall from the Introduction that the 2010 horizontal merger guidelines of the DOJ and the FTC state that horizontal mergers are unlikely to have adverse competitive effects when the post-merger HHI is below $1,500$, but express concerns about horizontal merges when the post-merger HHI is above $2,500$. If $\eta(Q^*) = 2$, i.e., a 1% increase in output leads to a 2% increase in consumers' surplus (which, as I show below, holds for instance when demand is linear), then HHI below $1,500$ is associated with a ratio of at most $0.15 \times 2 = 0.3$ between producers' surplus and consumers' surplus (that is, consumers' surplus is at least 3.3 times larger than producers' surplus), while a value of HHI above $2,500$ is associated with a ratio of at least $0.25 \times 2 = 0.5$ between producers' surplus and consumers' surplus (consumers' surplus is at least twice as large as producers' surplus). Viewed in this way, one can infer that whenever $\eta(Q^*) = 2$, the DOJ and the FTC have competitive concerns when consumers' surplus is at most twice as large as producers' surplus (1/0.5), but not when consumers' surplus is at least 3.3 times as large as producers' surplus (1/0.3). These ratios are even larger when $\eta(Q^*) < 2$, but smaller when $\eta(Q^*) > 2$.[6]

Turning to changes in HHI, equation (5) implies that, if we hold $\eta(Q^*)$ constant, an increase in HHI is associated with an increase in $\frac{PS^*}{CS^*}$. But since HHI is endogenous, changes in HHI due to demand or cost shocks or changes in the number of firms following entry, exit, or mergers, are also likely to affect $\eta(Q^*)$ both directly and indirectly (through their effect on Q^*). Consequently, equation (5) implies that an increase in H is associated with an increase in $\frac{PS^*}{CS^*}$ only when $H\eta(Q^*)$ moves in the same direction as H.

The next lemma, whose proof is in the Appendix, shows that a large family of demand functions has a constant $\eta(Q^*)$, in which case $H\eta(Q^*)$ surely increases with H.

Lemma 1: *An inverse demand function exhibits a constant elasticity of consumers' surplus if and*

[6] It should be noted that since the value of $\eta(Q^*)$ is likely to vary across industries (and over time), two industries with the same HHI could feature very different divisions of surplus between firms' owners and consumers. This suggests in turn that setting thresholds based only on HHI may be associated with very different distributional outcomes across industries.

only if it can be expressed as:

$$p(Q) = A - bQ^{\delta}, \tag{6}$$

where $A \geq 0$ *and* $b\delta > 0$. *The resulting elasticity of consumers' surplus,* $\eta(Q^*)$, *is then constant and given by* $1 + \delta$.

Anderson and Renault (2007) refer to demand functions that satisfy (6) as ρ-linear.[7] The family of ρ-linear demand functions, which exhibit a constant $\eta(Q^*)$, is quite broad. It includes as special cases linear demand functions when $A, b > 0$ and $\delta = 1$;[8] log-linear inverse demand functions when $A = \widetilde{A} + \frac{\widetilde{b}}{\delta}$, $b = \frac{\widetilde{b}}{\delta}$, and $\delta \to 0$, in which case the inverse demand function becomes $p = \widetilde{A} - \widetilde{b}\ln(Q)$;[9] and iso-elastic demand functions when $A = 0$, and $b, \delta < 0$, in which case the inverse demand function becomes $p = -bQ^{\delta}$. In the latter case, $-\frac{1}{\delta}$ represents the (constant) elasticity of demand. To ensure that the monopoly price is bounded from above, it must be that $\delta \in (-1, 0)$.

Together with Proposition 1, Lemma 1 implies the following Corollary:

Corollary 1: *In an n firms Cournot model where firms have (possibly different) constant marginal costs and the inverse demand function is given by (6),*

$$H = \frac{PS^*}{(1+\delta)\, CS^*}, \tag{7}$$

where $\delta \in (-1, 0)$ *if demand has a constant elasticity* $-\frac{1}{\delta}$, $\delta = 0$ *if the inverse demand function is log-linear, and* $\delta = 1$ *if demand is linear.*

Corollary 1 implies that when demand is ρ-linear, there is a constant relationship between HHI and the ratio of producers' surplus to consumers' surplus: every 100 points increase in HHI is associated with an increase in producers' surplus relative to consumers' surplus by $0.1(1+\delta)$. Interestingly, Bulow and Klemperer show that when demand is given by (6), $1 + \delta$ is the inverse of the cost pass-through rate, i.e., the rate at which a monopoly with a constant marginal cost k will

[7] A function f is called ρ-linear if f^{ρ} is linear. The name derives from the fact that the associated demand function, $Q(p) = \left(\frac{A-p}{b}\right)^{\frac{1}{\delta}}$, is ρ-linear when $\rho = \frac{1}{\delta}$. The family of ρ-linear demand functions was first used by Bulow and Pfleiderer (1983).

[8] By Lemma 1 then, when demand is linear, $\eta(Q^*) = 1 + \delta = 2$, which is the value I used above to illustrate Proposition 1.

[9] To see this, note that since $A = \widetilde{A} + \frac{\widetilde{b}}{\delta}$ and $b = \frac{\widetilde{b}}{\delta}$, the inverse demand function is $p(Q) = \widetilde{A} + \frac{\widetilde{b}\left(1-Q^{\delta}\right)}{\delta}$. Using L'Hôpital's rule, $\lim_{\delta \to 0}\left(\widetilde{A} + \frac{\widetilde{b}\left(1-Q^{\delta}\right)}{\delta}\right) = \lim_{\delta \to 0}\left(\widetilde{A} - \frac{\widetilde{b}Q^{\delta}\ln(Q)}{1}\right) = \widetilde{A} - \widetilde{b}\ln(Q)$.

raise its price in response to an increase in k.[10] $1+\delta$ is also related to the curvature of the demand function, $\sigma \equiv -\frac{p''(Q)Q}{p'(Q)}$, as $1+\delta = 2-\sigma$.

An interesting implication of Corollary 1 is that for a given value of HHI, $\frac{PS^*}{CS^*}$ is lower when demand is iso-elastic than when it is log-linear and is more than twice lower than when demand is linear (in the linear demand case, $\frac{PS^*}{CS^*}$ is twice as large as in the log-linear case). Consequently, all else equal, the distribution of social surplus is less favorable to firms' owners and more favorable to consumers when demand is iso-elastic than when the inverse demand function is log-linear, and it is most favorable to firm's owners and least favorable to consumers when demand is linear.

Another interesting implication of Corollary 1 is that when the inverse demand function is linear or log-linear, knowing H is sufficient to determine how social surplus is distributed between firms' owners and consumers. In either case, there is no need to know any other parameter to determine the relationship between HHI and $\frac{PS^*}{CS^*}$. In particular, consumers' surplus is $1/H$ times larger than producers' surplus when demand is log-linear and half that when demand is linear. For example, when HHI is $1,500$, consumers' surplus is 6.7 times larger than producers' surplus when demand is log-linear $(1/0.15)$ and 3.3 times larger when demand is linear; when HHI is $2,500$, consumers' surplus is 4 times larger than producers' surplus when demand is log-linear $(1/0.25)$ and twice larger when demand is linear; and when HHI is $5,000$, consumers' surplus is twice as larger as producers' surplus when demand is log-linear $(1/0.5)$ and equal in size to producers' surplus when demand is linear.

When the inverse demand function is iso-elastic, the relationship between HHI and $\frac{PS^*}{CS^*}$ depends on the parameter δ, which is the inverse of the elasticity of demand. Equation (7) shows that holding HHI constant, the distribution of social welfare becomes more favorable to firms' owners and less favorable to consumers as demand becomes more elastic and δ increases from -1 to 0. To illustrate, suppose that $p(Q) = Q^{\delta}$ (i.e., $A=0$ and $b=-1$), where $\delta \in (-1,0)$ (the elasticity of demand, $-\frac{1}{\delta}$, grows in this case from 1 to ∞) Assuming that firms have the same marginal cost, k, the profit of each firm i is $\pi_i = \left(Q^{\delta} - k\right) q_i$.[11] In a symmetric Nash equilibrium,

[10] The monopoly output of a firm with a constant marginal cost k is given by the following first-order condition: $p + p'Q - k = 0$. Fully differentiating this condition, yields $\frac{dQ}{dk} = \frac{1}{2p' + p"Q}$. Hence, $\frac{dp}{dk} = \frac{p'}{2p' + p"Q}$. When the inverse demand function if given by (7), the last equation becomes $\frac{dp}{dk} = \frac{1}{1+\delta}$.

[11] π_i is concave in q_i, because $\pi_i'' = \delta Q^{\delta-2}\left(2Q + (\delta-1)\, q_i\right) < 0$, where the inequality follows because $\delta \in (-1,0)$ and $2Q + (\delta-1)\, q_i > 0$.

the quantity of each firm is

$$q^* = \frac{1}{n}\left(\frac{kn}{\delta+n}\right)^{\frac{1}{\delta}}.$$

Aggregate output then is $Q^* = \left(\frac{kn}{\delta+n}\right)^{\frac{1}{\delta}}$ and the equilibrium price is $p(Q^*) = \frac{kn}{\delta+n}$. Hence, producers' surplus and consumers' surplus are given by

$$PS^* = \left(\frac{kn}{\delta+n} - k\right)\left(\frac{kn}{\delta+n}\right)^{\frac{1}{\delta}} = -\delta n^{\frac{1}{\delta}}\left(\frac{k}{\delta+n}\right)^{\frac{1+\delta}{\delta}},$$

and

$$CS^* = \int_0^{Q^*} z^\delta dz - (Q^*)^\delta Q^* = -\frac{\delta (Q^*)^{1+\delta}}{1+\delta} = -\frac{\delta}{1+\delta}\left(\frac{kn}{\delta+n}\right)^{\frac{1+\delta}{\delta}}.$$

Although PS^* and CS^* may either increase or decrease with δ depending on the parameter values,[12] their ratio is linearly increasing with δ:

$$\frac{PS^*}{CS^*} = \frac{-\delta n^{\frac{1}{\delta}}\left(\frac{k}{\delta+n}\right)^{\frac{1+\delta}{\delta}}}{-\frac{\delta}{1+\delta}\left(\frac{kn}{\delta+n}\right)^{\frac{1+\delta}{\delta}}} = \frac{1+\delta}{n}. \tag{8}$$

Noting that since firms are symmetric, $H = \frac{1}{n}$, (8) coincides with (7). It shows that $\frac{PS^*}{CS^*}$ is higher as demand becomes more elastic and δ increases from -1 towards 0.

Finally, one may wonder what happens if the demand function is not ρ-linear so that $\eta(Q^*)$ is no longer constant. One example for such a case is the Logit demand function

$$Q(p) = \frac{\beta e^{-\lambda p}}{1+\beta e^{-\lambda p}}.$$

The associated inverse demand function is given by

$$p(Q) = \frac{\ln(\beta) + \ln(1-Q) - \ln(Q)}{\lambda}.$$

In the Appendix, I show that in this case, $\eta'(Q^*) > 0$ for $Q^* < 0.483$ and $\eta'(Q^*) < 0$ for $0.483 < Q^* < 1$. Now suppose that a demand or a cost shock or a change in the number of firms causes an increase in H. Sufficient conditions for this change to also cause an increase in $H\eta(Q^*)$ are (i) Q^* decreases but is still above 0.483 (so $\eta'(Q^*) < 0$), or (ii) Q^* increases but is still below 0.483 (so $\eta'(Q^*) > 0$). When these conditions hold, the increase in H will be also associated with an increase in producers' surplus relative to consumers' surplus.

[12] For instance, when $n = 10$ and $k = 0.8$, PS^* is decreasing with δ for $-1 < \delta < -0.2185$ and increasing for $-0.2185 < \delta < 0$, while CS^* is decreasing with δ for $-1 < \delta < -0.1803$ and increasing for $-0.1803 < \delta < 0$.

3 Differentiated products

I now show that the key insight from the Cournot model carries over to models of differentiated products, provided that the demand system is linear. To this end, suppose that the n firms produce differentiated products and each firm i is facing an inverse demand function $p_i(q_1, \ldots, q_n)$ and has a cost function $c_i(q_i) = F_i + k_i q_i$, where $k_i < p_i(q_1, \ldots, q_n)$ when $q_i = 0$. The profit of each firm i is given by

$$\pi_i = (p_i(q_1, \ldots, q_n) - k_i) q_i - F_i.$$

An interior Nash equilibrium when firms compete by setting quantities is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:

$$p_i(q_1, \ldots, q_n) + \frac{\partial p_i(q_1, \ldots, q_n)}{\partial q_i} q_i - k_i = 0, \qquad i = 1, 2, , \ldots, n. \tag{9}$$

Since in a Nash equilibrium, $p_i^* - k_i = -\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i} q_i^*$, the equilibrium producer surplus of each firm i is

$$PS_i^* = (p_i^* - k_i) q_i^* = -\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i} (q_i^*)^2. \tag{10}$$

Hence, $PS^* \equiv \sum_{i=1}^n PS_i^*$ is a function of $\sum_{i=1}^n (q^*)^2$, which is the denominator of HHI, only if $\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i}$ is constant. This holds however only if $p_i(q_1, \ldots, q_n)$ is linear in q_i.

Under price competition, the profit of each firm i is

$$\pi_i = (p_i - k_i) q_i(p_1, \ldots, p_n) - F_i,$$

where $q_i(p_1, \ldots, p_n)$ is the demand that firm i is facing. An interior Nash equilibrium is now a vector $(p_1^*, \ldots, p_n^*)$ that solves the following system of first-order conditions,

$$q_i(p_1, \ldots, p_n) + \frac{\partial q_i(p_1, \ldots, p_n)}{\partial p_i} (p_i - k_i) = 0, \qquad i = 1, 2, , \ldots, n. \tag{11}$$

Since in a Nash equilibrium, $p_i^* - k_i = -\frac{q_i^*}{\frac{\partial q_i(p_1, \ldots, p_n)}{\partial p_i}}$, the equilibrium producer surplus of each firm i is

$$PS_i^* = (p_i^* - k_i) q_i^* = \frac{(q_i^*)^2}{-\frac{\partial q_i(p_1, \ldots, p_n)}{\partial p_i}}. \tag{12}$$

Once again, $PS^* \equiv \sum_{i=1}^n PS_i^*$ is a function of $\sum_{i=1}^n (q^*)^2$ only if $q_i(p_1, \ldots, p_n)$ is linear in p_i.

In what follows, I will therefore assume that the inverse demand system is linear and given by

$$p_i = A_i - \beta q_i - \gamma \sum_{j \neq i}^n q_j, \qquad i = 1, 2, , \ldots, n, \tag{13}$$

where $A_1, \ldots, A_n$ and β, are positive parameters, and $0 < \gamma < \beta$ is a measure of the degree of product differentiation, with lower values of γ representing a larger degree of differentiation.[13] This inverse demand system corresponds to the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, but if $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, where $\tau > 0$, it corresponds to the Shubik and Levitan (1980) specification.[14] In the latter case, the parameter τ reflects the degree of product differentiation, with lower values of τ representing a larger degree of differentiation.

3.1 Quantity competition

With quantity competition, PS_i^* is given by (10), where (13) implies that $\frac{\partial p_i\left(q_1^*,\ldots,q_n^*\right)}{\partial q_i} = -\beta$ for all i. Hence $PS^* \equiv \sum_{i=1}^n PS_i^* = \beta \sum_{i=1}^n (q^*)^2$. In the Appendix, I show that, evaluated at the equilibrium quantities, consumers' surplus is given by

$$CS^* = \frac{(\beta-\gamma)\sum_{i=1}^n (q_i^*)^2 + \gamma (Q^*)^2}{2}. \tag{14}$$

Noting that $\sum_{i=1}^n (q^*)^2 = \frac{PS^*}{\beta}$ and substituting for Q^* from (14) into (4) and rearranging, yields the following result:

Proposition 2: *In an n firms differentiated products oligopoly with quantity competition, where firms have (possibly different) constant marginal costs and face the linear inverse demand system (13), HHI is given by*

$$H = \frac{\sum_{i=1}^n \left(\frac{PS_i^*}{\beta}\right)}{\frac{2CS^*}{\gamma} - \frac{\beta-\gamma}{\gamma}\sum_{i=1}^n \left(\frac{PS_i^*}{\beta}\right)} = \frac{\frac{PS^*}{CS^*}}{\frac{2\beta}{\gamma} - \frac{\beta-\gamma}{\gamma}\frac{PS^*}{CS^*}}. \tag{15}$$

In the Shubik-Levitan case, where $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, HHI is given by

$$H = \frac{\frac{PS^*}{CS^*}}{\frac{2(n+\tau)}{\tau} - \frac{n}{\tau}\frac{PS^*}{CS^*}}. \tag{16}$$

Proposition 2 implies that, similarly to the Cournot case, HHI is positively related to $\frac{PS^*}{CS^*}$. Notice from (15) that as $\gamma \to \beta$ (products become homogeneous), the right-hand side of (15) approaches $\frac{1}{2}\frac{PS^*}{CS^*}$, which by equation (7), is the value of H under Cournot competition when demand is linear (in which case $\delta = 1$).[15]

[13] Obviously, γ cannot be too low relative to β otherwise the products are not in the same market in which case HHI becomes meaningless.

[14] A third notable example for a differentiated products oligopoly model with linear demands is the Vickery-Salop circular city model (Vickery 1964 and Salop 1979)).

[15] In the Appendix I also verify that when the n firms are symmetric and have the same marginal cost, the right-hand sides of (7) and (16) are equal to $1/n$ which is the value of H when firms are symmetric.

3.2 Price competition

To study the relationship between HHI and $\frac{PS^*}{CS^*}$ under price competition, we first need to invert the inverse demand system (13). In the Appendix, I show that the demand system associated with (13) is given by:

$$q_i = \kappa\left(A_i - p_i\right) - \rho\sum_{j\neq i}^{n}\left(A_i - p_j\right), \qquad i = 1, 2, , \ldots, n, \tag{17}$$

where

$$\kappa \equiv \frac{\beta + (n-2)\gamma}{(\beta-\gamma)(\beta+(n-1)\gamma)}, \qquad \rho \equiv \frac{\gamma}{(\beta-\gamma)(\beta+(n-1)\gamma)}.$$

Now, PS_i^* is given by (12), where (17) implies that $-\frac{\partial q_i(p_1,\ldots,p_n)}{\partial p_i} = \kappa$ for all i. Hence $PS^* \equiv \sum_{i=1}^{n} PS_i^* = \frac{1}{\kappa}\sum_{i=1}^{n}(q^*)^2$. Substituting in (4), noting that consumers' surplus is still given by (14), and rearranging, yields the following result:

Proposition 3: *In an n firms differentiated products oligopoly with price competition, where firms have (possibly different) constant marginal costs and face a linear demand system (17), HHI is given by*

$$H = \frac{\gamma\sum_{i=1}^{n}\left(\kappa \times PS_i^*\right)}{2CS^* - (\beta-\gamma)\sum_{i=1}^{n}\left(\kappa \times PS_i^*\right)} = \frac{\frac{PS^*}{CS^*}}{\frac{\beta-\gamma}{\gamma}\left(\frac{2(\beta+(n-1)\gamma)}{\beta+(n-2)\gamma} - \frac{PS^*}{CS^*}\right)}. \tag{18}$$

In the Shubik-Levitan case, where $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, HHI is given by

$$H = \frac{\frac{PS^*}{CS^*}}{\frac{n}{\tau}\left(\frac{2n(1+\tau)}{n(1+\tau)-\tau} - \frac{PS^*}{CS^*}\right)}. \tag{19}$$

Proposition 3 shows that under price competition, HHI is also positively related to $\frac{PS^*}{CS^*}$, similarly to the quantity competition case.

3.3 The normative implications of changes in HHI

To examine the normative implications of changes in HHI, it is important to bear in mind that HHI is endogenous, and hence it is not immediately obvious from Propositions 2 and 3 that an increase in HHI is necessarily associated with an increase in $\frac{PS^*}{CS^*}$, because the factors that cause an increase in HHI may also affect the relationship between HHI and $\frac{PS^*}{CS^*}$. To explore this issue further, note first that equations (15)-(16) and (18)-(19) are independent of the demand parameters $A_1, \ldots, A_n$, and the cost parameters $k_1, \ldots, k_n$. Hence, an increase in HHI due to changes in these parameters will be also associated with an increase in $\frac{PS^*}{CS^*}$, regardless of whether firms engage in quantity or price competition.

Turning to the demand parameters β and γ, I first prove in the Appendix that HHI is increasing with $\frac{\gamma}{\beta}$ under both quantity and price competition. To examine how these changes affect the relationship between HHI and $\frac{PS^*}{CS^*}$, it is useful to rewrite (15) as $\frac{PS^*}{CS^*} = \frac{2}{1+\frac{\gamma}{\beta}\left(\frac{1}{H}-1\right)}$ and (18) as $\frac{PS^*}{CS^*} = \frac{2H\left(1-\frac{\gamma}{\beta}\right)\left(1+(n-1)\frac{\gamma}{\beta}\right)}{\left(H\left(1-\frac{\gamma}{\beta}\right)+\frac{\gamma}{\beta}\right)\left(1+(n-2)\frac{\gamma}{\beta}\right)}$. The right-hand sides of the two equations are increasing with H and decreasing with $\frac{\gamma}{\beta}$.[16] Hence, an increase in HHI due to an increase in β or a decrease in γ (the n products become more differentiated) will be also associated with an increase in $\frac{PS^*}{CS^*}$ under both quantity and price competition.

In the Shubik-Levitan case, where $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, the relationship between HHI and $\frac{PS^*}{CS^*}$ depends on the demand parameter, τ. Since $\frac{\gamma}{\beta} = \frac{\tau}{n+\tau}$ is increasing with τ, HHI which is increasing with $\frac{\gamma}{\beta}$ under both quantity and price competition, is also increasing with τ. Rewriting (16) as $\frac{PS^*}{CS^*} = \frac{2(n+\tau)}{\frac{\tau}{H}+n}$ and (19) as $\frac{PS^*}{CS^*} = \frac{2n^2(1+\tau)}{\left(n+\frac{\tau}{H}\right)(n(1+\tau)-\tau)}$, and noting that the right-hand sides of the two equations are increasing with H and decreasing with τ, it follows that an increase in HHI due to a decrease in τ (the n products become more differentiated) is associated with an increase in $\frac{PS^*}{CS^*}$ under both quantity and price competition.[17]

Finally, I consider an increase in HHI due to a change in the number of firms, n. Equation (15) is independent of n, implying that under quantity competition, an increase in HHI due to a change in n will be also associated with an increase in $\frac{PS^*}{CS^*}$. Equation (18) depends on n, but rewriting it as $\frac{PS^*}{CS^*} = \frac{2H\left(1-\frac{\gamma}{\beta}\right)\left(1+(n-1)\frac{\gamma}{\beta}\right)}{\left(H\left(1-\frac{\gamma}{\beta}\right)+\frac{\gamma}{\beta}\right)\left(1+(n-2)\frac{\gamma}{\beta}\right)}$ and noting that the right-hand side is increasing with H and decreasing with n, it follows that an increase in H due to a decrease in n is associated with an increase in $\frac{PS^*}{CS^*}$.

As for the Shubik-Levitan case, recall that (16) and (19) can be rewritten as $\frac{PS^*}{CS^*} = \frac{2(n+\tau)}{\frac{\tau}{H}+n}$ and $\frac{PS^*}{CS^*} = \frac{2n^2(1+\tau)}{\left(\frac{\tau}{H}+n\right)(n(1+\tau)-\tau)}$. Since the right-hand sides of the two equations are increasing with both n and H, an increase in H due to a decrease in n can be associated with either an increase or decrease in $\frac{PS^*}{CS^*}$. However, in the symmetric case where $H = \frac{1}{n}$, the two equations become $\frac{PS^*}{CS^*} = \frac{2\left(1+\frac{\tau}{n}\right)}{1+\tau}$ and $\frac{PS^*}{CS^*} = \frac{2}{(1+\tau)\left(1+\tau-\frac{\tau}{n}\right)}$, and are clearly decreasing with n. Hence, an increase in H due to a decrease in n is associated with an increase in $\frac{PS^*}{CS^*}$.

[16] The derivative of the right-hand side of the latter equation with respect to $\frac{\gamma}{\beta}$ is $-\frac{2H\left[1-H\left(1-\frac{\gamma}{\beta}\right)^2+2(n-2)\frac{\gamma}{\beta}+\left(3-3n+n^2\right)\left(\frac{\gamma}{\beta}\right)^2\right]}{\left(H\left(1-\frac{\gamma}{\beta}\right)+\frac{\gamma}{\beta}\right)^2\left(1+(n-2)\frac{\gamma}{\beta}\right)^2}$, which is negative since $n > 1 > H$ and since $3 - 3n + n^2 > 0$ for all n.

[17] The derivatives of the right-hand sides of the equations with respect to τ are $-\frac{2n(1-H)}{H\left(n+\frac{\tau}{H}\right)^2} < 0$ and $-\frac{2n^2[\tau(2+\tau)(n-1)+n(1-H)]}{H\left(n+\frac{\tau}{H}\right)^2(n(1+\tau)-\tau)^2} < 0$, where the inequalities follow because $n > 1 > H$.

Proposition 4: *An increase in HHI is associated with an increase in* $\frac{PS^*}{CS^*}$ *in all cases, except when demand is given by the Shubik-Levitan specification and the increase in HHI is due to a decrease in the number of firms, in which case the associated change in* $\frac{PS^*}{CS^*}$ *is in general ambiguous. However, in the symmetric case where* $H = \frac{1}{n}$*, an increase in HHI due to a decrease in the number of firms is associated with an increase in* $\frac{PS^*}{CS^*}$*.*

Proposition 4 shows that when demand is given by the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, an increase in HHI is always associated with an increase in $\frac{PS^*}{CS^*}$, no matter whether the increase in HHI is driven by demand or cost shocks or a decrease in the number of firms. Hence, higher values of HHI are associated with distributions of social surplus that are more favorable to firms' owners and less favorable to consumers. The same conclusion also holds when demand is given by the Shubik and Levitan (1980) specification, provided that the increase in HHI is due to demand or cost shocks, or when firms are symmetric and the increase in HHI is due to a decrease in the number of firms.

4 Conclusion

I showed that in either the Cournot model with linear cost functions or a differentiated products model with linear demand and linear cost functions, HHI is an increasing function of the ratio of producers' surplus and consumers' surplus and hence reflects the division of the social surplus between firms' owners and consumers. In particular, higher values of HHI are associated with distributions of social surplus that are more favorable to firms' owners and less favorable to consumers. This result implies that HHI is directly related to measures of welfare and hence has an intuitive normative interpretation.

5 Appendix

Following are the proof of Lemma 1; derivation of $\eta(Q^*)$ in the Logit demand case; derivation of the demand system and consumers' surplus in the product differentiation case; a proof that in the product differentiation case, HHI is increasing with the demand parameter β and decreasing with the demand parameter γ; and checking that in the product differentiation case, $H = \frac{1}{n}$ when firms are symmetric.

Proof of Lemma 1: First I show that a constant elasticity of consumers' surplus, $\overline{\eta}$, implies a constant pass-through rate, $\frac{1}{\overline{\eta}}$. To this end, suppose that $\eta(Q^*) \equiv \frac{Q^*(CS^*)'}{CS^*} = \overline{\eta}$ for all Q^*. Since $\eta(Q^*)$ is constant,

$$\eta'(Q^*) = \frac{\left((CS^*)' + Q^*(CS^*)''\right)CS^* - Q^*\left((CS^*)'\right)^2}{(CS^*)^2} = 0,$$

which implies that

$$\left((CS^*)' + Q^*(CS^*)''\right)CS^* = Q^*\left((CS^*)'\right)^2.$$

Rewriting the equality,

$$\frac{(CS^*)' + Q^*(CS^*)''}{(CS^*)'} = \frac{Q^*(CS^*)'}{CS^*} = \overline{\eta}.$$

Recalling that $(CS^*)' = -p'(Q^*)Q^*$ and noting that $(CS^*)'' = -p'(Q^*) - p''(Q^*)Q^*$, yields

$$\begin{aligned} \overline{\eta} &= \frac{(CS^*)' + Q^*(CS^*)''}{(CS^*)'} \\ &= \frac{-p'(Q^*)Q^* - Q^*\left(p'(Q^*) + p''(Q^*)Q^*\right)}{-p'(Q^*)Q^*} \\ &= \frac{2p'(Q^*) + p''(Q^*)Q^*}{p'(Q^*)}. \end{aligned} \tag{20}$$

The last expression however is just the inverse of the cost pass-through rate. To see why, note that if the market is served by a monopoly with a constant marginal cost k, the profit-maximizing output, Q^*, is implicitly defined by the first-order condition $p(Q^*) + p'(Q^*)Q^* - k = 0$. Fully differentiating the first-order condition with respect to Q^* and k and rearranging, yields

$$\frac{\partial Q^*}{\partial k} = \frac{1}{2p'(Q^*) + p''(Q^*)Q^*}.$$

Hence, the cost pass-through rate is

$$p'(k) \equiv p'(Q^*)\frac{\partial Q^*}{\partial k} = \frac{p'(Q^*)}{2p'(Q^*) + p''(Q^*)Q^*}. \tag{21}$$

Together with (20), this implies that $p'(k) = \frac{1}{\overline{\eta}}$.

Next, I show that a constant cost pass-through rate, $\frac{1}{\overline{\eta}}$, implies a constant elasticity of consumers' surplus, $\overline{\eta}$. Bulow and Pfleiderer (1983) prove that an inverse demand function exhibits a constant cost pass-through rate if and only if it is represented by (6). The resulting cost pass-through rate is

$$p'(k) = \frac{-b\delta Q^{\delta-1}}{-2b\delta Q^{\delta-1} - b\delta(\delta-1)Q^{\delta-2}Q} = \frac{1}{1+\delta},$$

and consumers' surplus is given by

$$\begin{aligned} CS^* &= \int_0^{Q^*}\left(A - bz^{\delta}\right)dz - \left(A - b\left(Q^*\right)^{\delta}\right)Q^* \\ &= AQ^* - \frac{b\left(Q^*\right)^{1+\delta}}{1+\delta} - \left(A - b\left(Q^*\right)^{\delta}\right)Q^* \qquad (22) \\ &= \frac{\delta b\left(Q^*\right)^{1+\delta}}{1+\delta}. \end{aligned}$$

The elasticity of CS^* with respect to output in this case is

$$\eta(Q^*) = \left(\frac{\delta b\left(Q^*\right)^{1+\delta}}{\delta+1}\right)' \frac{Q^*}{\frac{\delta b(Q^*)^{\delta+1}}{\delta+1}} = 1+\delta.$$

Hence, if $p'(k) = \frac{1}{1+\delta} \equiv \frac{1}{\overline{\eta}}$, then $\eta(Q^*) = 1+\delta = \overline{\eta}$. ■

$\boldsymbol{\eta}(Q^*)$ in the Logit demand case: Given the inverse demand function $p(Q) = \frac{\ln(\beta)+\ln(1-Q)-\ln(Q)}{\lambda}$, consumers' surplus is given by

$$\begin{aligned} CS^* &= \frac{1}{\lambda}\int_0^{Q^*}\left(\ln(\beta) + \ln(1-z) - \ln(z)\right)dz - \frac{Q^*\left(\ln(\beta) + \ln(1-Q^*) - \ln(Q^*)\right)}{\lambda} \\ &= \frac{\left(Q^*\ln(\beta) - \left((1-Q^*)\ln(1-Q^*) + Q^*\right) - \left(Q^*\ln(Q^*) - Q^*\right)\right)}{\lambda} - \frac{Q^*\left(\ln(\beta) + \ln(1-Q^*) - \ln(Q^*)\right)}{\lambda} \\ &= -\frac{\ln(1-Q^*)}{\lambda}. \end{aligned}$$

The elasticity of CS^* is

$$\eta(Q^*) = \frac{Q^*\left(CS^*\right)'}{CS^*} = -\frac{Q^*}{(1-Q^*)\ln(1-Q^*)}.$$

Note that $\eta(Q^*)$ depends directly only on Q^*, but not on the parameters β and λ. Differentiating yields,

$$\begin{aligned} \eta'(Q^*) &= -\frac{(1-Q^*)\ln(1-Q^*) + \ln(1-Q^*) + 1}{\left((1-Q^*)\ln(1-Q^*)\right)^2} \\ &= -\frac{(2-Q^*)\ln(1-Q^*) + 1}{\left((1-Q^*)\ln(1-Q^*)\right)^2}. \end{aligned}$$

The sign of $\eta'(Q^*)$ depends on the sign of the numerator which is positive for $Q^* < 0.483$ and negative for $0.483 < Q^* < 1$. ■

The demand system in the product differentiation case: From (13) it follows that

$$q_i = \frac{1}{\beta}\left(A_i - p_i - \gamma \sum_{j\neq i}^{n} q_j\right).$$

Adding $-\frac{\gamma}{\beta}q_i$ to both sides of the equation, recalling that $Q = \sum_{i=1}^{n} q_i$, and rearranging, yields

$$q_i - \frac{\gamma}{\beta}q_i = \frac{1}{\beta}\left(A_i - p_i - \gamma \sum_{j\neq i}^{n} q_j - \gamma q_i\right), \qquad \Rightarrow \qquad q_i = \frac{A_i - p_i - \gamma Q}{\beta - \gamma}.$$

Summing over all firms and solving for Q, yields

$$Q = \frac{\sum_{i=1}^{n}(A_i - p_i) - \gamma n Q}{\beta - \gamma} \qquad \Rightarrow \qquad Q = \frac{\sum_{i=1}^{n}(A_i - p_i)}{\beta + (n-1)\gamma}.$$

Substituting for Q in q_i and rearranging, yields (17). ■

Consumers' surplus in the product differentiation case with linear demands: Starting with the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, the demand system is derived from the preferences of a representative consumer, whose utility function is quadratic:

$$u(q_1, \ldots, q_n) = \sum_{i=1}^{n} A_i q_i - \frac{\beta \sum_{i=1}^{n} q_i^2 + \gamma \sum_{i=1}^{n} \sum_{j\neq i}^{n} q_i q_j}{2} + m, \tag{23}$$

where m is income spent on all other goods, $A_1, \ldots, A_n$ and β, are positive utility parameters, and $0 < \gamma < \beta$. Maximizing $u(q_1, \ldots, q_n)$ subject to a budget constraint, $\sum_{i=1}^{n} p_i q_i + m = I$, where p_i is the prices of good i, and I is income, yields the system of inverse demand functions (13).

To express consumers' surplus, note first that the utility function of the representative consumer can now be written as:

$$\begin{aligned} u(q_1, \ldots, q_n) &= \sum_{i=1}^{n} A_i q_i - \frac{(\beta - \gamma)\sum_{i=1}^{n} q_i^2 + \gamma \sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j}{2} + m \\ &= \sum_{i=1}^{n} A_i q_i - \frac{(\beta - \gamma)\sum_{i=1}^{n} q_i^2 + \gamma Q^2}{2} + m, \end{aligned}$$

where the last equality follows since $\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j = \left(\sum_{j=1}^{n} q_i\right)^2 = Q^2$. Substituting for m from the budget constraint into (23) and using (13), consumers' surplus is given by

$$
\begin{aligned}
CS\left(q_1,\ldots,q_n\right) &= \sum_{i=1}^{n} A_i q_i - \frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma Q^2}{2} - \sum_{i=1}^{n} p_i q_i \\
&= \sum_{i=1}^{n} A_i q_i - \frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^* + \gamma Q^2}{2} - \sum_{i=1}^{n}\left(A_i - \beta q_i - \gamma \sum_{j\neq i}^{n} q_j\right) q_i \\
&= -\frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma Q^2}{2} + (\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma \sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \\
&= \frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma Q^2}{2}.
\end{aligned}
$$

Evaluating at the equilibrium quantities, yields $CS^* \equiv CS\left(q_1^*,\ldots,q_n^*\right)$, given by (14).

The Shubik-Levitan (1980) demand system is derived similarly, except that now $\beta = \left(\frac{n+\tau}{1+\tau}\right)$ and $\gamma = \frac{\tau}{1+\tau}$. Given these parameter values, consumers' surplus at the equilibrium quantities is

$$
CS^* \equiv CS\left(q_1^*,\ldots,q_n^*\right) = \frac{n\sum_{i=1}^{n}\left(q_i^*\right)^2 + \tau\left(Q^*\right)^2}{2\left(1+\tau\right)}.
$$

■

HHI in the product differentiation case is increasing with β and decreasing with γ: I begin by considering quantity competition. Using (13), the profit of each firm i is

$$
\pi_i = \left(A_i - \beta q_i - \gamma \sum_{j\neq i}^{n} q_j - k_i\right) q_i.
$$

An interior Nash equilibrium when firms set quantities is a vector $(q_1^*,\ldots,q_n^*)$ that solves the following system of first-order conditions:

$$
A_i - 2\beta q_i - \gamma \sum_{j\neq i}^{n} q_j - k_i = 0, \qquad i = 1,2,,\ldots,n.
$$

Adding and subtracting γq_i from the left-hand side of the equation, recalling that $Q = \sum_{i=1}^{n} q_i$, and solving for q_i, yields the best-response function of each firm i against the aggregate quantity Q (which includes q_i):[18]

$$
q_i = \frac{A_i - k_i - \gamma Q}{2\beta - \gamma}.
$$

Summing over all $i = 1,2,,\ldots,n$, and solving for Q, yields

$$
Q^* = \frac{\sum_{i=1}^{n}\left(A_i - k_i\right)}{2\beta + (n-1)\gamma}.
$$

[18] The best-response of a firm against aggregate output was called by Selten (1973) the "fitting-in function."

Substituting Q^* in the best-response functions, yields

$$q_i^* = \frac{\left(2\beta + (n-1)\gamma\right)(A_i - k_i) - \gamma \sum_{i=1}^n (A_i - k_i)}{(2\beta - \gamma)\left(2\beta + (n-1)\gamma\right)}, \qquad i = 1, 2, , \ldots, n.$$

Given q_i^* and Q^*, the market share of each firm i, $s_i^* = \frac{q_i^*}{Q^*}$, is given by

$$\begin{aligned}
s_i^* &= \frac{\left(2\beta + (n-1)\gamma\right)(A_i - k_i) - \gamma \sum_{i=1}^n (A_i - k_i)}{(2\beta - \gamma)\sum_{i=1}^n (A_i - k_i)} \\
&= \frac{\left(2 + (n-1)\frac{\gamma}{\beta}\right)(A_i - k_i) - \frac{\gamma}{\beta}\sum_{i=1}^n (A_i - k_i)}{\left(2 - \frac{\gamma}{\beta}\right)\sum_{i=1}^n (A_i - k_i)} \\
&= \frac{\left(2 + (n-1)\frac{\gamma}{\beta}\right)(A_i - k_i)}{\left(2 - \frac{\gamma}{\beta}\right)\sum_{i=1}^n (A_i - k_i)} - \frac{\frac{\gamma}{\beta}}{2 - \frac{\gamma}{\beta}}.
\end{aligned}$$

It is straightforward to check that $\sum_{i=1}^n s_i^* = 1$ and that under symmetry where $A_i = A$ and $k_i = k$ for all i, $s_i^* = \frac{1}{n}$. Given the market shares, HHI is given by

$$H = \sum_{i=1}^n \underbrace{\left(\frac{\left(2 + (n-1)\frac{\gamma}{\beta}\right)(A_i - k_i)}{\left(2 - \frac{\gamma}{\beta}\right)\sum_{i=1}^n (A_i - k_i)} - \frac{\frac{\gamma}{\beta}}{2 - \frac{\gamma}{\beta}}\right)}_{s_i^*}^2$$

Note that HHI depends on $\frac{\gamma}{\beta}$ rather than separately on β and γ. Differentiating HHI with respect to $\frac{\gamma}{\beta}$,

$$\begin{aligned}
\frac{\partial H}{\partial\left(\frac{\gamma}{\beta}\right)} &= 2\sum_{i=1}^n s_i^* \left(\frac{\left((n-1)\left(2 - \frac{\gamma}{\beta}\right) + 2 + (n-1)\frac{\gamma}{\beta}\right)(A_i - k_i)}{\left(2 - \frac{\gamma}{\beta}\right)^2 \sum_{i=1}^n (A_i - k_i)} - \frac{2}{\left(2 - \frac{\gamma}{\beta}\right)^2}\right) \\
&= \frac{4n}{\left(2 - \frac{\gamma}{\beta}\right)^2}\sum_{i=1}^n s_i^* \left(\frac{A_i - k_i}{\sum_{i=1}^n (A_i - k_i)} - \frac{1}{n}\right) \\
&= \frac{4n}{\left(2 - \frac{\gamma}{\beta}\right)^2}\left(\frac{\sum_{i=1}^n s_i^* (A_i - k_i)}{\sum_{i=1}^n (A_i - k_i)} - \frac{1}{n}\right).
\end{aligned}$$

To determine the sign of the inequality, note that the series $\frac{A_1 - k_1}{\sum_{i=1}^n (A_i - k_i)}, \ldots, \frac{A_n - k_n}{\sum_{i=1}^n (A_i - k_i)}$ can be ordered from large to small: $\frac{A_1 - k_1}{\sum_{i=1}^n (A_i - k_i)} \geq \ldots \geq \frac{A_n - k_n}{\sum_{i=1}^n (A_i - k_i)}$. Since the market share of each firm i is increasing with $A_i - k_i$, it also follows that $s_1^* \geq \ldots \geq s_n^*$. By Chebyshev's sum inequality then, $\frac{1}{n}\sum_{i=1}^n s_i^* \times (A_i - k_i) \geq \left(\frac{1}{n}\sum_{i=1}^n (A_i - k_i)\right)\left(\frac{1}{n}\sum_{i=1}^n s_i\right)$. Noting that $\sum_{i=1}^n s_i^* = 1$, it follows that $\frac{\sum_{i=1}^n s_i^* (A_i - k_i)}{\sum_{i=1}^n (A_i - k_i)} \geq \frac{1}{n}$. Hence, the derivative is nonnegative and is strictly positive when firms are not symmetric.

Next, I turn to price competition. Using (17), the profit of each firm i is

$$\pi_i = \left(\kappa\left(A_i - p_i\right) - \rho \sum_{j \neq i}^{n} \left(A_i - p_j\right)\right)\left(p_i - k_i\right).$$

An interior Nash equilibrium when firms set prices is a vector $(p_1^*, \ldots, p_n^*)$ that solves the following system of first-order conditions:

$$\kappa\left(A_i - p_i\right) - \rho \sum_{j \neq i}^{n} \left(A_i - p_j\right) - \kappa\left(p_i - k_i\right) = 0, \qquad i = 1, 2, , \ldots, n.$$

Adding and subtracting $\rho\left(A_i - p_i\right)$ from the left-hand side of the equation and reorganizing terms,

$$\kappa\left(A_i + k_i\right) + \rho A_i - \left(2\kappa + \rho\right) p_i - \rho \sum_{i=1}^{n} \left(A_i - p_i\right) = 0, \qquad i = 1, 2, , \ldots, n.$$

Solving for p_i, yields the best-response function of each firm i against the sum of the prices of all firms, $\sum_{i=1}^{n} p_i$ (including p_i):

$$p_i = \frac{\kappa\left(A_i + k_i\right) + \rho A_i - \rho \sum_{i=1}^{n} A_i + \rho \sum_{i=1}^{n} p_i}{2\kappa + \rho}.$$

Summing over all $i = 1, 2, , \ldots, n$, and solving for $\sum_{i=1}^{n} p_i$, yields

$$\sum_{i=1}^{n} p_i^* = \frac{\kappa \sum_{i=1}^{n} \left(A_i + k_i\right) - \rho\left(n-1\right) \sum_{i=1}^{n} A_i}{2\kappa - \left(n-1\right)\rho}.$$

Substituting $\sum_{i=1}^{n} p_i^*$ in the best-response functions, yields

$$\begin{aligned}
p_i^* &= \frac{\kappa\left(A_i + k_i\right) + \rho A_i - \rho \sum_{i=1}^{n} A_i + \rho \frac{\kappa \sum_{i=1}^{n}\left(A_i + k_i\right) - \rho(n-1) \sum_{i=1}^{n} A_i}{2\kappa - (n-1)\rho}}{2\kappa + \rho} \\
&= \frac{\left(\kappa + \rho\right) A_i + \kappa k_i}{2\kappa + \rho} - \frac{\rho\kappa \sum_{i=1}^{n}\left(A_i - k_i\right)}{\left(2\kappa + \rho\right)\left(2\kappa - \left(n-1\right)\rho\right)}.
\end{aligned}$$

Given p_i^* and $\sum_{i=1}^{n} p_i^*$, the quantity of firm i is given by

$$\begin{aligned}
q_i^* &= \kappa\left(A_i - p_i^*\right) - \rho \sum_{j \neq i}^{n} \left(A_i - p_j^*\right) \\
&= \left(\kappa + \rho\right)\left(A_i - p_i^*\right) - \rho \sum_{i=1}^{n} A_i + \rho \sum_{i=1}^{n} p_i^* \\
&= \left(\kappa + \rho\right)\left(A_i - \frac{\left(\kappa + \rho\right) A_i + \kappa k_i}{2\kappa + \rho} + \frac{\kappa\rho \sum_{i=1}^{n}\left(A_i - k_i\right)}{\left(2\kappa + \rho\right)\left(2\kappa - \left(n-1\right)\rho\right)}\right) \\
&\quad - \rho \sum_{i=1}^{n} A_i + \rho \frac{\kappa \sum_{i=1}^{n}\left(A_i + k_i\right) - \rho\left(n-1\right) \sum_{i=1}^{n} A_i}{2\kappa - \left(n-1\right)\rho} \\
&= \left(\kappa + \rho\right)\left(\frac{\kappa\left(A_i - k_i\right)}{2\kappa + \rho} + \frac{\kappa\rho \sum_{i=1}^{n}\left(A_i - k_i\right)}{\left(2\kappa + \rho\right)\left(2\kappa - \left(n-1\right)\rho\right)}\right) - \frac{\kappa\rho \sum_{i=1}^{n}\left(A_i - k_i\right)}{2\kappa - \left(n-1\right)\rho} \\
&= \frac{\kappa\left(\kappa + \rho\right)\left(A_i - k_i\right)}{2\kappa + \rho} - \frac{\kappa^2\rho \sum_{i=1}^{n}\left(A_i - k_i\right)}{\left(2\kappa + \rho\right)\left(2\kappa - \left(n-1\right)\rho\right)}.
\end{aligned}$$

Summing over all firms,

$$
\begin{aligned}
Q^* &= \frac{\kappa(\kappa+\rho)\sum_{i=1}^{n}(A_i-k_i)}{2\kappa+\rho} - \frac{n\kappa^2\rho\sum_{i=1}^{n}(A_i-k_i)}{(2\kappa+\rho)(2\kappa-(n-1)\rho)} \\
&= \frac{\kappa((\kappa+\rho)(2\kappa-(n-1)\rho)-n\kappa\rho)\sum_{i=1}^{n}(A_i-k_i)}{(2\kappa+\rho)(2\kappa-(n-1)\rho)} \\
&= \frac{\kappa(\kappa-(n-1)\rho)\sum_{i=1}^{n}(A_i-k_i)}{2\kappa-(n-1)\rho}.
\end{aligned}
$$

The market share of each firm i is then

$$
\begin{aligned}
s_i^* &= \frac{\frac{\kappa(\kappa+\rho)(A_i-k_i)}{2\kappa+\rho} - \frac{\kappa^2\rho\sum_{i=1}^{n}(A_i-k_i)}{(2\kappa+\rho)(2\kappa-(n-1)\rho)}}{\frac{\kappa(\kappa-(n-1)\rho)\sum_{i=1}^{n}(A_i-k_i)}{2\kappa-(n-1)\rho}} \\
&= \frac{(\kappa+\rho)(2\kappa-(n-1)\rho)(A_i-k_i)-\kappa\rho\sum_{i=1}^{n}(A_i-k_i)}{(2\kappa+\rho)(\kappa-(n-1)\rho)\sum_{i=1}^{n}(A_i-k_i)} \\
&= \frac{\left(1+\frac{\rho}{\kappa}\right)\left(2-(n-1)\frac{\rho}{\kappa}\right)(A_i-k_i)-\frac{\rho}{\kappa}\sum_{i=1}^{n}(A_i-k_i)}{\left(2+\frac{\rho}{\kappa}\right)\left(1-(n-1)\frac{\rho}{\kappa}\right)\sum_{i=1}^{n}(A_i-k_i)} \\
&= \frac{\left(1+\frac{\rho}{\kappa}\right)\left(2-(n-1)\frac{\rho}{\kappa}\right)(A_i-k_i)}{\left(2+\frac{\rho}{\kappa}\right)\left(1-(n-1)\frac{\rho}{\kappa}\right)\sum_{i=1}^{n}(A_i-k_i)} - \frac{\frac{\rho}{\kappa}}{\left(2+\frac{\rho}{\kappa}\right)\left(1-(n-1)\frac{\rho}{\kappa}\right)}.
\end{aligned}
$$

The market shares then, and hence HHI, depend only on $\frac{\rho}{\kappa}$, but not separately on the parameters κ and ρ. HHI is then given by

$$
H = \sum_{i=1}^{n}\left[\frac{\left(1+\frac{\rho}{\kappa}\right)\left(2-(n-1)\frac{\rho}{\kappa}\right)(A_i-k_i)}{\left(2+\frac{\rho}{\kappa}\right)\left(1-(n-1)\frac{\rho}{\kappa}\right)\sum_{i=1}^{n}(A_i-k_i)} - \frac{\frac{\rho}{\kappa}}{\left(2+\frac{\rho}{\kappa}\right)\left(1-(n-1)\frac{\rho}{\kappa}\right)}\right]^2.
$$

Differentiating HHI with respect to $\frac{\rho}{\kappa}$,

$$
\begin{aligned}
\frac{\partial H}{\partial\left(\frac{\rho}{\kappa}\right)} &= \frac{\left(2+(n-1)\left(\frac{\rho}{\kappa}\right)^2\right)n}{\left(2+\frac{\rho}{\kappa}\right)^2\left(1-(n-1)\frac{\rho}{\kappa}\right)^2}\sum_{i=1}^{n}s_i^*\left(\frac{A_i-k_i}{\sum_{i=1}^{n}(A_i-k_i)}-\frac{1}{n}\right) \\
&= \frac{\left(2+(n-1)\left(\frac{\rho}{\kappa}\right)^2\right)n}{\left(2+\frac{\rho}{\kappa}\right)^2\left(1-(n-1)\frac{\rho}{\kappa}\right)^2}\left(\frac{\sum_{i=1}^{n}s_i^*(A_i-k_i)}{\sum_{i=1}^{n}(A_i-k_i)}-\frac{1}{n}\right) \geq 0,
\end{aligned}
$$

where the inequality follows by Chebyshev's sum inequality. Since $\frac{\rho}{\kappa} = \frac{\gamma}{\beta+(n-2)\gamma} = \frac{\frac{\gamma}{\beta}}{1+(n-2)\frac{\gamma}{\beta}}$ is increasing with $\frac{\gamma}{\beta}$, so is HHI. ■

HHI in the product differentiation case with symmetric firms: When the n firms are identical, $q_i^* = q^*$ for all i. Substituting in (14), yields

$$
CS^* = \frac{(\beta-\gamma)n(q_i^*)^2+\gamma(nq^*)^2}{2} = \frac{n(\beta+\gamma(n-1))(q_i^*)^2}{2}.
$$

Noting that under symmetry and quantity competition, $PS^* = \beta n (q^*)^2$, it follows that $\frac{PS^*}{CS^*} = \frac{2\beta}{\beta+\gamma(n-1)}$. Substituting in (15) and simplifying, yields

$$H = \frac{\frac{2\beta}{\beta+\gamma(n-1)}}{\frac{2\beta}{\gamma} - \frac{\beta-\gamma}{\gamma}\frac{2\beta}{\beta+\gamma(n-1)}} = \frac{1}{n}.$$

Under price competition, $PS^* = \frac{n(q^*)^2}{\kappa}$. Hence, $\frac{PS^*}{CS^*} = \frac{\frac{n(q^*)^2}{\kappa}}{\frac{n(\beta+\gamma(n-1))\left(q_i^*\right)^2}{2}} = \frac{2}{\kappa(\beta+\gamma(n-1))}$. Substituting in (18) and simplifying, yields

$$H = \frac{\frac{2}{\kappa(\beta+\gamma(n-1))}}{\frac{2}{\kappa\gamma} - \frac{\beta-\gamma}{\gamma}\frac{2}{\kappa(\beta+\gamma(n-1))}} = \frac{1}{n}.$$

Since the result is independent of the parameters β and γ, it also holds for the Shubik and Levitan specification. ■

6 References

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